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Gauss-Manin system and the flat coordinate system

(Connection with the expansion of the solutions at ∞)

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In this note, we consider the differential system of Gauss-Manin associated with a versal deformation of a simple singularity. As to such a Gauss-Manin system, it is known by K.Saito [4] that the parameter space, say S, has a canonical linear structure. A flat coordinate system is a "linear" coordinate system with respect to the linear structure of S. The purpose of this note is to give an explicit construction of the flat coordinate system by means of the expansion of the solutions near a point at infinity.

§1. Gauss-Manin system and the flat coordinate system.

We begin with a review of the notion of a flat coordinate system. Let $f(x) = f(x_1, \ldots, x_n)$ be one of the following canonical forms of simple isolated singularities:

$$(1.1) \quad \begin{cases} A_\ell : x_1^{\ell+1} \ (n=1), & D_\ell : x_1^{\ell-1} + x_1 x_2^2 \ (n=2), \\ E_6 : x_1^4 + x_2^3, \ E_7 : x_1^3 + x_1 x_2^3, & E_8 : x_1^5 + x_2^3 \ (n=2). \end{cases}$$

Here, the subscript ℓ or 6,7,8 stands for the Milnor number $\mu = \mu(f)$ of f at $x=0$:

$$(1.2) \qquad \mu = \dim_{\mathbb{C}} \mathcal{O}_{\mathbb{C}^n,0} / (\partial_x f), \quad (\partial_x f) = (\partial_{x_1} f, \ldots, \partial_{x_n} f).$$

Note that f is weighted homogeneous of degree 1 with respect to a unique weight $\rho = (\rho_1, \ldots, \rho_n)$, where ρ_i are positive rational numbers. In our case, the quotient ring $\mathcal{O}_{\mathbb{C}^n,0}/(\partial_x f)$ has a $\mathbb{C}$-basis consisting of monomials. To fix the ideas, we take a set $N \subset \mathbb{N}^n$ of multi-indices as follows, so that the residue classes of $x^\nu = x_1^{\nu_1} \ldots x_n^{\nu_n}$ $(\nu = (\nu_1, \ldots, \nu_n) \in N)$ form a basis of the ring $\mathcal{O}_{\mathbb{C}^n,0}/(\partial_x f)$.

$$(1.3)\begin{cases} \text{(I) Case of } A_\ell, E_6, E_8 \ (f = x_1^{p_1} + \ldots + x_n^{p_n}) : \\ \qquad N = \left\{ \nu \in \mathbb{N}^n;\ 0 \leq \nu_i \leq p_i - 2 \ (1 \leq i \leq n) \right\}. \\ \text{(II) Case of } D_\ell, E_7 \ (f = x_1^{p_1} + x_1 x_2^{p_2}) : \\ \qquad N = \left\{ \nu \in \mathbb{N}^2;\ 0 \leq \nu_1 \leq p_1 - 1,\ 0 \leq \nu_2 \leq p_2 - 2 \right\} \cup \left\{ (0, p_2 - 1) \right\}. \end{cases}$$

Let $S = \mathbb{C}^\mu$ be the complex affine μ-space with coordinate system $t = (t_\nu)_{\nu \in N}$, which we regard as the space of deformation parameters. We take a versal deformation $F = F(t,x)$ of $f = f(x)$ defined by

$$(1.4) \qquad F(t,x) = f(x) + \sum_{\nu \in N} t_\nu x^\nu .$$

The deformation F is versal in the sense that the residue classes of $\partial F/\partial t_\nu |_{t=0}$ $(\nu \in N)$ form a basis of $\mathcal{O}_{\mathbb{C}^n,0}/(\partial_x f)$. Note again that F is weighted homogeneous of degree 1 with respect to a unique weight (r, ρ) of (t,x), $r = (r_\nu)_{\nu \in N}$, and that all r_ν $(\nu \in N)$ are positive rational numbers as an effect of simpleness of the singularity f. We pay a special attention to the parameter t_0. Let $T = \mathbb{C}^{\mu-1}$ be the affine $(\mu-1)$-space with coordinate system $t^* = (t_\nu)_{\nu \in N^*}$, where $N^* = N \setminus \{0\}$, and $\pi : S \longrightarrow T$ the canonical projection. Then the projection π is determined by the vector field $D_0 = D_{t_0} = \partial/\partial t_0$ and the parameter t_0 gives the fiber coordinate of π.

The Gauss-Manin system associated with F, denoted by $\underline{H}_F$, is by definition the differential system on S to be satisfied by the integral

$$u(t) = \int \delta(F(t,x))\,dx\,,\quad dx = dx_1 \wedge \dots \wedge dx_n\,. \tag{1.5}$$

For each $\nu \in N$, define

$$u_\nu = \int \frac{\partial F}{\partial t_\nu}\,\delta(F)\,dx = \int x^\nu \delta(F)\,dx \quad (u_0 = u)\,. \tag{1.6}$$

Then one can represent the Gauss-Manin system as a differential system on S including the column vector $\vec{u} = (u_\nu)_{\nu \in N}$ as the unknown.

<u>Proposition 1</u>. <u>The Gauss-Manin system</u> $\underline{H}_F$ <u>has a finite presentation</u>

$$\left\{\begin{aligned} D_{t_0} t_0 \vec{u} &= (A_0(t^*) D_{t_0} + A_1(t^*))\,\vec{u}\,,\\ D_{t_\nu} \vec{u} &= (B_0^\nu(t^*) D_{t_0} + B_1^\nu(t^*))\,\vec{u} \quad (\nu \in N^*)\,, \end{aligned}\right. \tag{1.7}$$

<u>where</u> $A_i,\ B_i^\nu \in M(\mu; \mathbb{C}[t^*])$ $(i = 0,1;\ \nu \in N^*)$.

Let us recall here the Gauss-Manin system $\underline{H}_{f+t_0}$. For each $\nu \in N$, define $w_\nu = \int x^\nu \delta(f + t_0)\,dx$ and consider the column vector $\vec{w} = (w_\nu)_{\nu \in N}$. Then the Gauss-Manin system $\underline{H}_{f+t_0}$ is given by

$$D_{t_0} t_0 \vec{w} = -\Lambda \vec{w}\,,\quad -\Lambda = \operatorname{diag}(\varepsilon_\nu;\ \nu \in N)\,. \tag{1.8}$$

Note that the <u>exponents</u> ε_ν $(\nu \in N)$ of f are determined by the formulas

$$L_{\theta_x}(x^{\nu}dx) = \varepsilon_{\nu} x^{\nu} dx \quad (\nu \in N), \tag{1.9}$$

where L_{θ_x} is the Lie derivative with respect to the Euler vector field $\theta_x = \sum_{i=1}^{n} \rho_i x_i D_{x_i}$. Comparing the differential system (1.7) with (1.8), one sees that

$$A_0|_{t^*=0} = 0 \quad \text{and} \quad A_1|_{t^*=0} = -\Lambda, \tag{1.10}$$

since $\vec{u}$ is a deformation of $\vec{w} = \vec{u}|_{t^*=0}$.

Now we recall the notion of a flat coordinate system of S. Let $s = (s_{\nu})_{\nu \in N}$ be a coordinate system of S relative to $\pi : S \longrightarrow T$. Namely, suppose that π is realized as the projection $s = (s_0, s^*) \longmapsto s^* = (s_{\nu})_{\nu \in N^*}$ and that $D_{s_0} = D_0$. For such a coordinate system $s = (s_{\nu})_{\nu \in N}$, we define

$$v_{\nu} = \int \frac{\partial F}{\partial s_{\nu}} \delta(F)\, dx \quad \text{for } \nu \in N. \tag{1.11}$$

Then, as to the differential system for the column vector $\vec{v} = (v_{\nu})_{\nu \in N}$, we have

<u>Theorem 2.</u> <u>There exists a coordinate system</u> $s = (s_{\nu})_{\nu \in N}$ <u>of</u> S <u>relative to</u> $\pi : S \longrightarrow T$ <u>such that the Gauss-Manin system</u> $\underline{H}_F$ <u>is represented in the form</u>

$$\begin{cases} D_{s_0} s_0 \vec{v} = (A(s^*) D_{s_0} - \Lambda) \vec{v}, \\ D_{s_{\nu}} \vec{v} = B^{\nu}(s^*) D_{s_0} \vec{v} \quad (\nu \in N^*), \end{cases} \tag{1.12}$$

<u>where</u> $A, B^{\nu} \in M(\mu; \mathbb{C}[s^*])$ $(\nu \in N^*)$. <u>Moreover,</u> <u>such a coordinate system</u> $s = (s_{\nu})_{\nu \in N}$ <u>is determined uniquely up to linear transformation</u>.

A coordinate system $s=(s_\nu)_{\nu\in N}$ relative to π is called a flat coordinate system if it has the property as stated in Theorem 2. At the same time, Theorem 2 says that the space S of parameters can be endowed with a linear structure. For an intrinsic formulation of the linear structure and the flat coordinate system, we refer the reader to K.Saito [4] or S.Ishiura-M.Noumi [2] .

§2. Expansion of the solutions near a point at infinity.

We keep the assumptions and the notations in §1. Here we explain how one can construct the flat coordinate system of S by means of the expansion of the solutions of $\underline{H}_F$ at infinity.

As to the differential system (1.7) of Proposition 1, we define the discriminant $\Delta=\Delta(t)$ by

$$\Delta(t)=\det(t_0-A_0(t^*)), \tag{2.1}$$

which is a monic polynomial of degree μ in t_0 with coefficients in $\mathbb{C}[t^*]$. Then the system (1.7) defines an integrable meromorphic connection over S with logarithmic poles along the discriminant set $D=\{\Delta=0\}$. Hence, one sees that the system (1.7) has a unique fundamental system $\Phi(t)$ of many-valued holomorphic solutions on $S\setminus D$ such that

$$\Phi\in GL(\mu;\mathcal{O}(\widetilde{S\setminus D})) \quad\text{and}\quad \Phi|_{t^*=0}=t_0^{-\Lambda-1}. \tag{2.2}$$

Now we take the compactification $\overline{S}=\mathbb{P}\times T$ of $S=\mathbb{C}\times T$ in the direction of t_0-axis. Then, it is easy to see that the discriminant set D is closed in $\overline{S}$ and does not intersect with the hyperplane

$\{t_0 = \infty\}$ at infinity. Moreover, the meromorphic connection in question has regular singularities along $\{t_0 = \infty\}$ in the naïve sense. So one sees that $\Phi(t)$ can be factorized in the form

$$\Phi(t) = \Psi(t)\, t_0^{-\Lambda-1} \quad \text{near} \quad (t_0, t^*) = (\infty, 0)\,, \tag{2.3}$$

where $\Psi(t) \in GL(\mu; \mathcal{O}_{\overline{S},(\infty,0)})$ and $\Psi|_{t^*=0} = 1$. (It is known that this kind of expression gives rise to a hypergeometric representation of the solutions in the case where $f = x_1^{p_1} + \ldots + x_n^{p_n}$ and $F = f + t_1 x_1 + \ldots + t_n x_n + t_0$.) We formulate this expression (2.3) in connection with the flat coordinate system.

<u>Proposistion 3</u>. (a) <u>There is a unique formal differential operator</u>

$$P(t^*, D_{t_0}) = \sum_{k=0}^{\infty} P_k(t^*) D_{t_0}^k\,, \quad P_k \in M(\mu; \mathbb{C}[t^*])\,, \tag{2.4}$$

<u>such that</u>

$$\Phi(t) = P(t^*, D_{t_0})\, t_0^{-\Lambda-1} \quad \text{near} \quad (t_0, t^*) = (\infty, 0)\,. \tag{2.5}$$

(b) <u>As to the operator</u> P <u>in (a)</u>, <u>define</u>

$$s_\nu = \delta_{0,\nu} t_0 + P_1(t^*)_{0,\nu}\,, \quad \underline{\text{for}}\ \nu \in N\,. \tag{2.6}$$

<u>Then</u>, $s = (s_\nu)_{\nu \in N}$ <u>gives a flat coordinate system of</u> S .

Remark that the 0-th row of the matrix P of operators represents the expansion of μ independent solutions of $u = \int \delta(F)\, dx$.

Now we mention how the matrix P can be determined explicitly. The first step is to solve a system of difference equations for a function $c : \mathbb{N}^n \longrightarrow \mathbb{C}$. To describe the difference system, we introduce some notations. We denote by T_i $(1 \leq i \leq n)$ the translation by one in the i-th direction. Namely, for a function $c : \mathbb{N}^n \longrightarrow \mathbb{C}$, we define a function $T_i c : \mathbb{N}^n \longrightarrow \mathbb{C}$ by

$$(2.7) \qquad (T_i c)(\alpha) = c(\alpha + 1_i) \quad \text{for } \alpha \in \mathbb{N}^n ,$$

where 1_i stands for the unit vector $(0,\ldots,\overset{i}{\check{1}},\ldots,0)$. Then the difference system to be solved is given by

$$(2.8) \qquad \alpha_i c(\alpha - 1_i) + f_i(T)c(\alpha) = 0 \quad (\alpha \in \mathbb{N}^n;\ 1 \leq i \leq n),$$

where $T = (T_1,\ldots,T_n)$ and $f_i = \partial_{x_i} f$ $(1 \leq i \leq n)$. One can see that the difference system (2.8) has μ independent solutions, denoted by c_ν $(\nu \in N)$, as follows.

<u>Case of</u> A_ℓ, E_6, E_8 where $f = x_1^{p_1} + \ldots + x_n^{p_n}$. We define a lattice L of $\mathbb{Z}^n$ by $L = \sum_{i=1}^n \mathbb{Z} p_i 1_i$. Then, for each $\nu \in N$, we define a function $c_\nu : \mathbb{N}^n \longrightarrow \mathbb{C}$ by

$$(2.9) \qquad c_\nu(\alpha) = \prod_{i=1}^n (-)^{k_i} \left(\frac{\nu_i + 1}{p_i}; k_i\right), \quad k_i = \frac{\alpha_i - \nu_i}{p_i},$$

if $\alpha \in (\nu + L) \cap \mathbb{N}^n$ and by $c_\nu(\alpha) = 0$ if $\alpha \in \mathbb{N}^n \setminus (\nu + L)$.

<u>Case of</u> D_ℓ, E_7 where $f = x_1^{p_1} + x_1 x_2^{p_2}$. We define a lattice L of $\mathbb{Z}^n$ by $L = \mathbb{Z} p_1 1_1 + \mathbb{Z}(p_2 1_2 + 1_1)$. Then, for each $\nu \in N$, we define a function $c_\nu : \mathbb{N}^2 \longrightarrow \mathbb{C}$ by

$$(2.10) \qquad c_\nu(\alpha) = (-)^{k_1}\left(\frac{\nu_1 + 1}{p_1} - \frac{\nu_2 + 1}{p_1 p_2}; k_1\right) . (-)^{k_2}\left(\frac{\nu_2 + 1}{p_2}; k_2\right)$$

with $k_1 = \dfrac{\alpha_1 - \nu_1}{p_1} - \dfrac{\alpha_2 - \nu_2}{p_1 p_2}$ and $k_2 = \dfrac{\alpha_2 - \nu_2}{p_2}$ if $\alpha \in (\nu + L) \cap \mathbb{N}^n$ and by $c_\nu(\alpha) = 0$ if $\alpha \in \mathbb{N}^2 \setminus (\nu + L)$.

In (2.9) and (2.10), the expression $(z;k)$ stands for the factorial function $\Gamma(z+k)/\Gamma(z)$. It is directly checked that the functions

c_ν $(\nu \in N)$ defined as above give μ independent solutions of (2.8). As the next step, we introduce a linear mapping $\ell : \mathbb{N}^{\mu-1} \longrightarrow \mathbb{N}^n$ by the formula

$$(2.11) \qquad \exp\left(\sum_{\nu \in N^*} t_\nu x^\nu\right) = \sum_{\alpha^* \in \mathbb{N}^{\mu-1}} \frac{t^{*\alpha^*}}{\alpha^*!} x^{\ell(\alpha^*)} .$$

In other words, the multi-index $\ell(\alpha^*) = (\ell_1(\alpha^*), \ldots, \ell_n(\alpha^*))$ is given by $\ell_i(\alpha^*) = \sum_{\nu \in N^*} \nu_i \alpha_\nu$ for $1 \leqslant i \leqslant n$.

<u>Theorem 4</u>. <u>By means of the functions</u> $c_\nu : \mathbb{N}^n \longrightarrow \mathbb{C}$ <u>and the linear mapping</u> $\ell : \mathbb{N}^{\mu-1} \longrightarrow \mathbb{N}^n$ <u>above, the operator</u> $P(t^*, D_{t_0})$ <u>of Proposition 3 is determined as follows</u>:

$$(2.12) \qquad P_k(t^*)_{\kappa,\nu} = \sum_{\alpha^* \in \mathbb{N}^{\mu-1}} c_\nu(\ell(\alpha^*) + \kappa) \frac{t^{*\alpha^*}}{\alpha^*!}$$

<u>for</u> $\kappa, \nu \in N$ <u>and</u> $k \in \mathbb{N}$.

<u>Corollary</u>. <u>The flat coordinate system of Proposition 3 can be represented by the formulas</u>

$$(2.13) \qquad s_\nu = \delta_{0,\nu} t_0 + \sum_{\alpha^* \in \mathbb{N}^{\mu-1}} c_\nu(\ell(\alpha^*)) \frac{t^{*\alpha^*}}{\alpha^*!} \quad (\nu \in N) .$$

Note that $P_k(t^*)_{\kappa,\nu}$ and so s_ν are weighted homogeneous polynomials. Details of the above argument are given in M.Noumi [3] .

§3. <u>A remark in the case of type</u> A_ℓ .

In the case of type A, it is known by S.Ishiura-M.Noumi [1] that the flat coordinate systems of type A_ℓ $(\ell \geqslant 1)$ can be obtained by a method of reduction from a countable sequence of weighted homogeneous polynomials. Here we remark that this sequence of polynomials eventually appears in the expansion (2.5) of the solutions of the Gauss-Manin system.

Let $\tilde{t} = (\tilde{t}_2, \tilde{t}_3, \ldots)$ be a sequence of countably many variables and consider the formal Laurent series

$$(3.1) \qquad f(x) = x + \tilde{t}_2 x^{-1} + \tilde{t}_3 x^{-2} + \ldots .$$

Then one can find a unique Laurent series

$$(3.2) \qquad g(y) = y - \tilde{s}_2 y^{-1} - \tilde{s}_3 y^{-2} + \ldots$$

such that $g \circ f(x) = x$ and $f \circ g(y) = y$, so that one obtains a new sequence $\tilde{s} = (\tilde{s}_2, \tilde{s}_3, \ldots)$ of countably many variables. Note that $\tilde{s}_2, \tilde{s}_3, \ldots$ are weighted homogeneous polynomials in $\tilde{t}_2, \tilde{t}_3, \ldots$ and vice versa. Moreover, the two sequences $\tilde{t} = (\tilde{t}_2, \tilde{t}_3, \ldots)$ and $\tilde{s} = (\tilde{s}_2, \tilde{s}_3, \ldots)$ are connected to each other by the formulas

$$(3.3) \qquad \begin{cases} \tilde{s}_k = \dfrac{1}{k-1} \operatorname{Res}(f(x)^k \, dx) & (k = 2,3,\ldots) \text{ and} \\ \tilde{t}_k = \dfrac{-1}{k-1} \operatorname{Res}(g(y)^k \, dy) & (k = 2,3,\ldots) . \end{cases}$$

There is reason to call this sequence $\tilde{s} = (\tilde{s}_2, \tilde{s}_3, \ldots)$ the "flat coordinate system" associated with $\tilde{t} = (\tilde{t}_2, \tilde{t}_3, \ldots)$.

Consider the versal deformation

$$(3.4) \qquad F(t,x) = x^n + t_2 x^{n-2} + t_3 x^{n-3} + \ldots + t_n , \quad t = (t_2, \ldots, t_n)$$

of the singularity $x^n = 0$ of type A_{n-1} (in a different way of indexing from that of preceding sections). Then one can take the fractional power $F^{1/n}$ in the form (3.1), so that $\tilde{t}_2, \tilde{t}_3, \ldots$ are weighted homogeneous polynomials in $t_2, \ldots, t_n$. (Among those,

$\tilde{t}_2,\ldots,\tilde{t}_n$ are algebraically independent.) Then the variables $\tilde{s}_2,\tilde{s}_3,\ldots$ in (3.2) are determined as polynomials in $t_2,\ldots,t_n$. Now, we define a sequence $s_2,s_3,\ldots$ of polynomials in $t_2,\ldots,t_n$ as follows:

$$s_k = n\,\tilde{s}_k \in \mathbb{C}[t] \quad \text{for } k = 2,3,\ldots . \tag{3.5}$$

Then one can prove that $s_2,\ldots,s_n$ are algebraically independent and that the coordinate system $(s_2,\ldots,s_n)$ coincides with the flat coordinate system given in Corollary of Theorem 4. It is by this reduction process that the flat coordinate systems of type A_ℓ are obtained from the sequence $\tilde{s} = (\tilde{s}_2,\tilde{s}_3,\ldots)$ of "flat variables". The sequence of "flat variables" also appears in the expansion of the solutions of the Gauss-Manin system $\underline{H}_F$ at infinity.

<u>Theorem 5. Let φ_k $(k = 2,\ldots,n)$ be the $n-1$ independent solutions of the Gauss-Manin system $\underline{H}_F$ that appear in the 0-th row of (2.5). Then we have</u>

$$\varphi_k = \delta_{k,n}\, s^{-\frac{n-1}{n}} + \sum_{r=1}^{\infty} (-)^r \left(\frac{k-1}{n}+r-1\right) s'_{(r-1)n+k}\, s_n^{-\frac{k-1}{n}-r}, \tag{3.6}$$

<u>where $s'_i = s_i|_{s_n=0}$ is the polynomial obtained from s_i by setting $s_n = 0$ for $i = 2,3,\ldots$.</u>

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